1. [Maximum points: 27]

In this problem you will investigate curves and families of curves which are said to be orthogonal.

Two curves are said to be *orthogonal* if at all points of intersection between the two curves the tangent lines to each curve are perpendicular to each other.

Let $f(x) = -\dfrac{2x + \sin 2x}{4}$ and $g(x) = \tan x$.

Figure 1 shows the graphs of $y = f(x)$ and $y = g(x)$ for $-2\pi \le x \le 2\pi$.

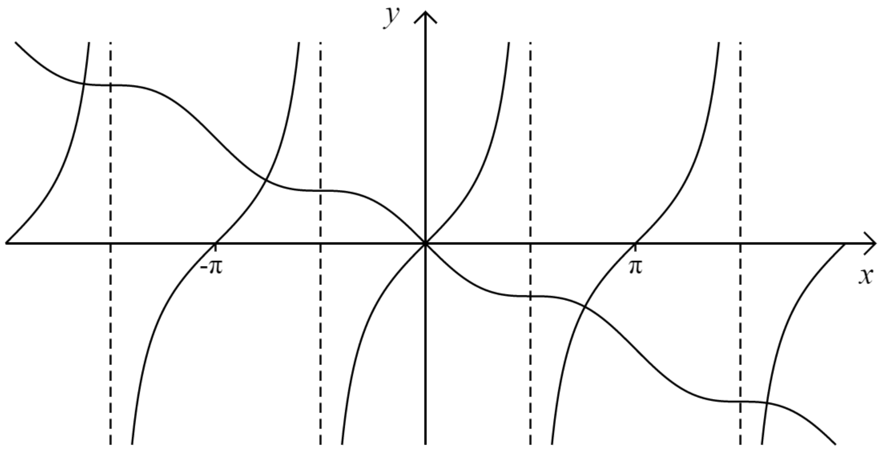


Figure 1

(a) Find [3]

(i) $f'(x)$

(ii) $g'(x)$

Let the x-coordinate of a point of intersection between the graphs of $y = f(x)$ and $y = g(x)$ be equal to a.

(b) At the point where $x = a$ write down an expression in terms of a for the gradient of the tangent to the graph of [2]

(i) $y = f(x)$

(ii) $y = g(x)$

(c) Hence show that the two curves are orthogonal. [3]

The diagram below shows the curve $x^2 + 4y^2 = 16$.

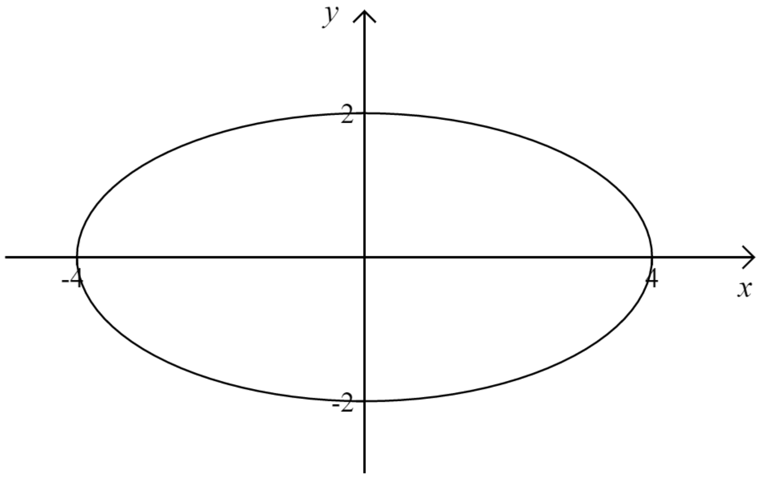


Figure 2

(d) Find $\frac{dy}{dx}$ in terms of x and y. [3]

(e) For a curve which is orthogonal to the curve in figure 2 show that at the points of intersection we need [2]

$$\frac{dy}{dx} = \frac{4y}{x}$$

(f) By treating the equation in part (e) as a separable differential equation find the equation of a function of y in terms of x that is orthogonal to the curve in figure 2. [3]

Two families of curves are said to be orthogonal if every curve in one family is orthogonal to every curve in the other family.

Consider the families of curves described by the equations $y = c\tan x$ and $y^2 = d - \sin^2 x$ where c and d are non-zero real numbers.

Figure 3 shows some curves from these families.

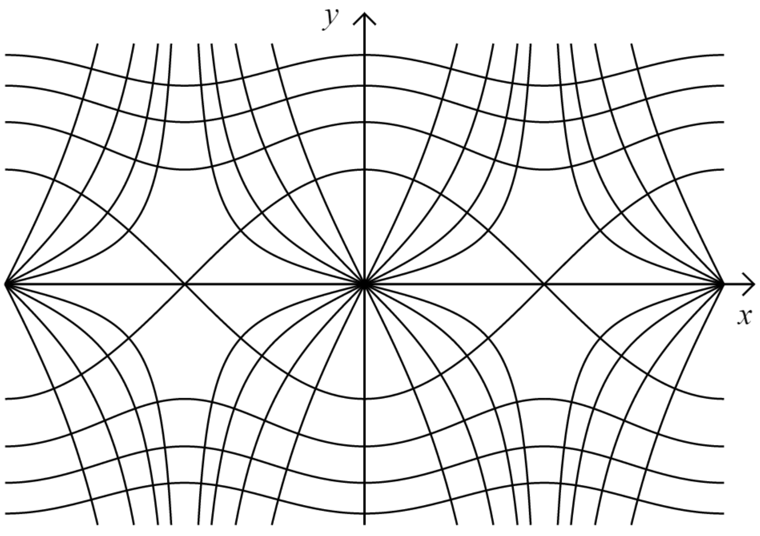


Figure 3

(g) Sketch the two curves when $c = 1$ and $d = 1/2$ for $-\pi \leq x \leq \pi$. [3]

(h) Differentiate $y = c\tan x$. [1]

(i) For the curve $y^2 = d - \sin^2 x$ find $\frac{dy}{dx}$ in terms of x and y. [3]

(j) Hence show that the two families of curves are orthogonal. [4]

2. [Maximum points: 28]

In this problem you will use linear regression techniques to find the equation of the curve of best fit for a non-linear set of data.

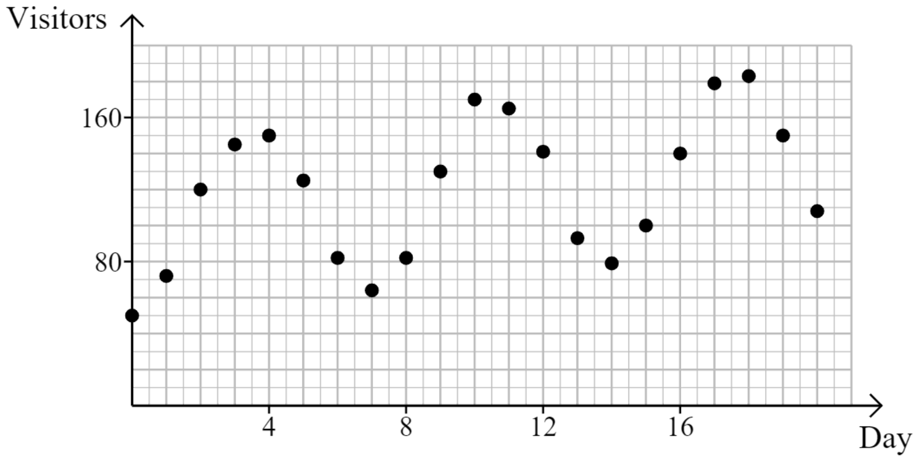
Let $f(x) = ax + b\cos(cx) + d$ and $g(x) = ax + d$ where $a, b, c, d \in \mathbb{R}$ and $c > 0$.

(a) Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes when $a = 1$, $b = 2$, $c = 3$ and $d = 3$ for $0 \le x \le 2\pi$. [4]

Paul creates a new website containing mathematics review problems for Diploma Programme students. The table below shows the number of visitors to the website for the first three weeks after the website is published. Day 0 represents a Saturday.

Day (t)	0	1	2	3	4	5	6	7	8	9	10
Visitors (v)	50	72	120	145	150	125	82	64	82	130	170
Day (t)	11	12	13	14	15	16	17	18	19	20	21
Visitors (v)	165	141	93	79	100	140	179	183	150	108	93

The scatter diagram below shows this data.



(b) Explain two possible reasons for the shape of the graph. [4]

Paul wishes to find the equation in the form $v = f(t)$ where f takes the form described above. To find the values of a and d he finds the equation of the least squares regression line for the data in the form $v = g(t)$ where g takes the form described above.

(c) Find the values of a and d. [2]

(d) Explain why $c = \frac{2\pi}{7}$. [3]

Paul decides to use least squares regression techniques to find the value of b manually. Let D_i represent the difference between the actual value of each v_i and the value of $f(t_i)$. Then let

$$D = \sum_{i=0}^{21} (D_i)^2 = \sum_{i=0}^{21} (v_i - at_i - b\cos(ct_i) - d)^2$$

where a, c and d take the values calculated in parts (c) and (d).

(e) By finding $\frac{dD}{db}$ show that to minimise the value of D we need [5]

$$-\sum_{i=0}^{21} v_i \cos(ct_i) + a\sum_{i=0}^{21} t_i \cos(ct_i) + b\sum_{i=0}^{21} \cos^2(ct_i) + d\sum_{i=0}^{21} \cos(ct_i) = 0$$

(f) Find the value of $\sum_{i=0}^{21} v_i \cos(ct_i)$. [4]

(g) Given that [2]

$$\sum_{i=0}^{21} t_i \cos(ct_i) = 10.5 \qquad \sum_{i=0}^{21} \cos^2(ct_i) = 11.5 \qquad \sum_{i=0}^{21} \cos(ct_i) = 1.00$$

find the value of b that minimises the value of D.

(h) Use your equation to estimate the amount of visitors the website will receive on day 60. [2]

(i) Comment on the accuracy of your answer to part (h). [2]

1. (a)

(i) $f'(x) = -\dfrac{2+2\cos 2x}{4} = -\dfrac{1+\cos 2x}{2}$ M1A1

(ii) $g'(x) = \sec^2 x$ A1

(b)

(i) $-\dfrac{1+\cos 2a}{2}$ A1

(ii) $\sec^2 a$ A1

(c) Use the double angle identity M1

$$-\frac{1+\cos 2a}{2} = -\cos^2 a$$ A1

We have

$$-\cos^2 a \times \frac{1}{\cos^2 a} = -1$$ A1

So the curves are orthogonal.

(d) Use implicit differentiation M1

$$2x + 8y\frac{dy}{dx} = 0$$ A1

So

$$\frac{dy}{dx} = -\frac{x}{4y}$$ A1

(e) We need

$$-\frac{x}{4y} \times \frac{dy}{dx} = -1$$ M1

So

$$\frac{dy}{dx} = \frac{4y}{x}$$ A1

(f) We have

$$\int \frac{1}{y}\,dy = 4\int \frac{1}{x}dx$$ M1

Giving

$$\ln|y| = 4\ln|x| + C$$ A1

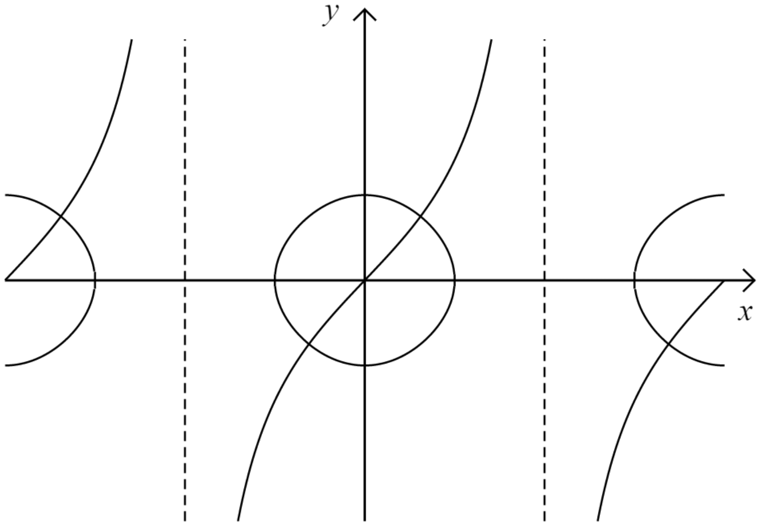
Therefore

$$y = Ax^4$$ A1

(g) The graph of $y = \tan x$ is drawn correctly. A1

The curve of $y^2 = 1/2 - \sin^2 x$ for $y \geq 0$ is drawn correctly. A1

The curve of $y^2 = 1/2 - \sin^2 x$ for $y < 0$ is drawn correctly. A1



(h) $\frac{dy}{dx} = c\sec^2 x$ A1

(i) Use implicit differentiation M1

$$2y\frac{dy}{dx} = -2\sin x\cos x$$ A1

So

$$\frac{dy}{dx} = -\frac{\sin x\cos x}{y}$$ A1

(j) At the points of intersection we have

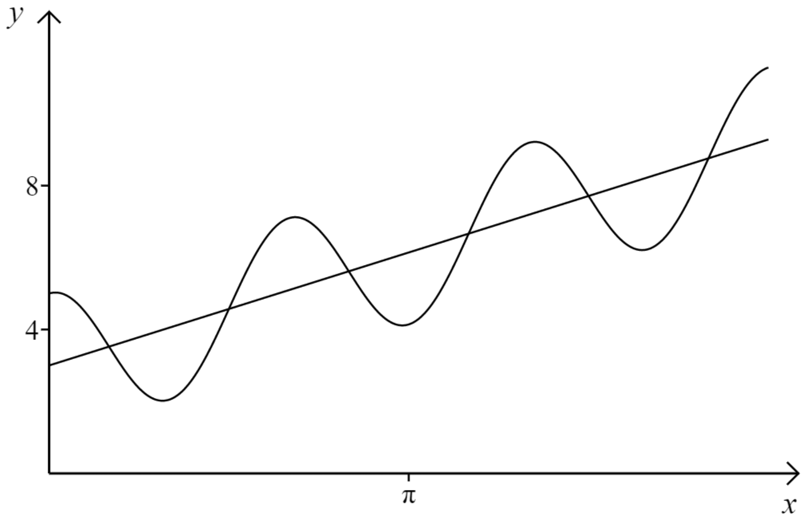
$$-\frac{\sin x\cos x}{y} \times c\sec^2 x = -\frac{c\tan x}{y} = -\frac{c\tan x}{c\tan x} = -1$$ M1A1A1

Since the product of the gradient is -1 the curves are perpendicular. R1

2. (a) The domain is $0 \leq x \leq 2\pi$. A1

The y-axis scale is appropriate. A1

The shape and position of each graph is approximately correct. A1A1



(b) The values of v are periodic because perhaps less people will use the website on a weekend when they dont have school. A1 R1

The values of v are gradually increasing perhaps because the popularity of the website is increasing through, for example, word of mouth or advertising. A1 R1

(c) $a = 2.01$ A1

$d = 98.0$ A1

(d) There are seven days in a week so the period of the cosine function should be 7. So R1

$$\frac{2\pi}{c} = 7$$ M1

Giving

$$c = \frac{2\pi}{7}$$ A1

(e) Use the chain rule M1

$$\frac{dD}{db}=\sum_{i=0}^{21}2(v_i-at_i-b\cos(ct_i)-d)(-\cos(ct_i))$$ A1

Expand

$$\frac{dD}{db}=-2\sum_{i=0}^{21}v_i\cos(ct_i)+2a\sum_{i=0}^{21}t_i\cos(ct_i)+2b\sum_{i=0}^{21}\cos^2(ct_i)+2d\sum_{i=0}^{21}\cos(ct_i)$$ A1

Set equal to 0 and simplify M1

$$-\sum_{i=0}^{21}v_i\cos(ct_i)+a\sum_{i=0}^{21}t_i\cos(ct_i)+b\sum_{i=0}^{21}\cos^2(ct_i)+d\sum_{i=0}^{21}\cos(ct_i)=0$$ A1

(f) The values of each $v_i\cos(ct_i)$ are A1A1A1

50	44.9	-26.7	-131	-135	-27.8	51.1	64	51.1	-28.9	-153
-149	-31.4	58.0	79	62.3	-31.2	-161	-165	-33.4	67.3	93

The sum is equal to -452. A1

(g) We have

$$452+2.01\times 10.5+11.5b+98.0\times 1.00=0$$ M1

So

$$b=-49.7$$ A1

(h) $2.01\times 60-49.7\cos(2\pi\times 60/7)+98.0=189$ M1A1

(i) This is not accurate since the value in (h) is an extrapolation. A1R1