1. [Maximum points: 4]

Hayley invests $2500 in an investment account pays 6.2% interest compounded monthly.

(a) Find the value of the investment after 1 year. [2]

A different investment account pays r % interest compounded annually.

(b) Find the minimum value of r so that it is more beneficial for Hayley to invest her money in this account. Write your answer to 2 decimal places. [2]

2. [Maximum points: 6]

The mass of apples produced on a farm is normally distributed with mean 150 g and standard deviation 4.2 g.

(a) An apple is selected at random. Find the probability its mass is greater than 160 g. [2]

(b) In a sample of 1000 apples find [4]

(i) the expected number of apples with a mass greater than 160 g

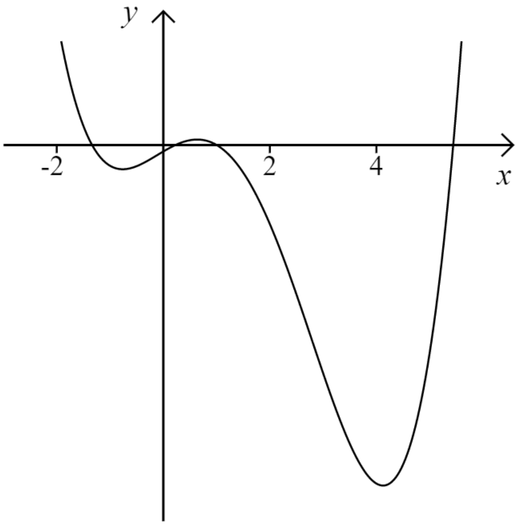
(ii) the probability more than 10 apples have a mass greater than 160 g

3. [Maximum points: 6]

Evaluate $\lim_{x \to 0} \frac{\cos x - 1}{\sin^2 x}$.

4. [Maximum points: 6]

Let $f(x) = 3x^4 - 16x^3 - 6x^2 + 24x - 5$. The diagram below shows the graph of $y = f(x)$.



(a) Find the coordinates of the turning points. Write your answers directly on the diagram above. [3]

(b) Find the restrictions on the value of k if the equation $f(x) = k$ has [3]

(i) 2 solutions

(ii) 4 solutions

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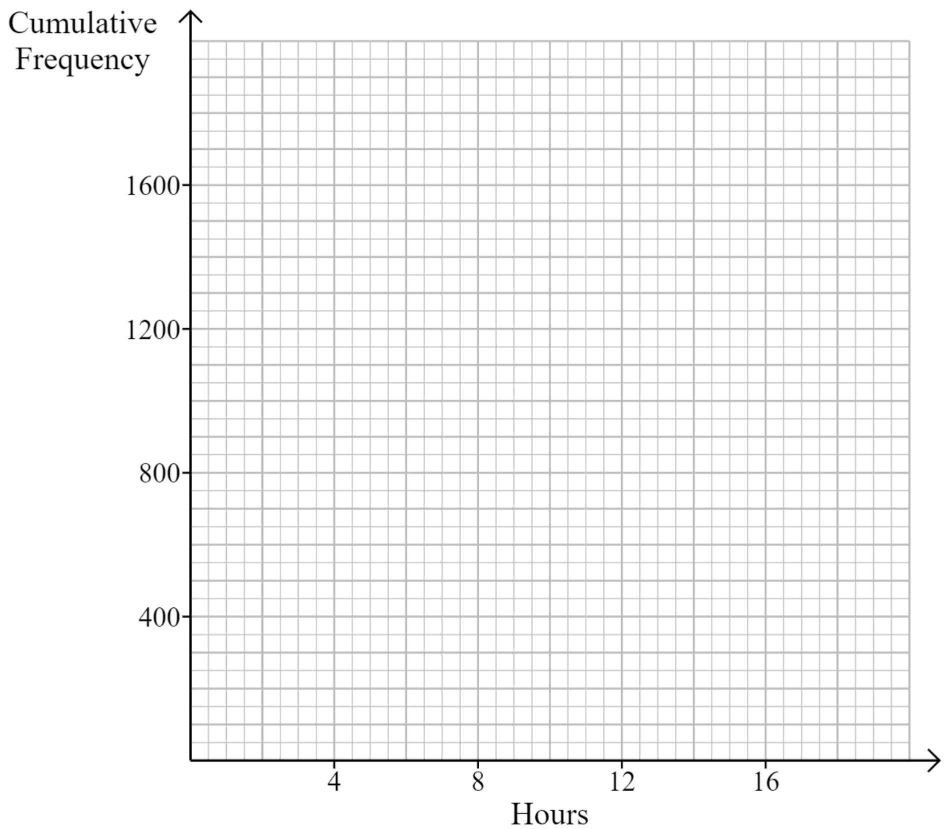
5. [Maximum points: 8]

All 2000 students in a school are surveyed to find how many hours they studied outside of class in one week. The results are shown in the table below.

Hours (h)	$0 < h \le 4$	$4 < h \le 8$	$8 < h \le 12$	$12 < h \le 16$	$16 < h \le 20$
Frequency	200	600	700	400	100
Cumulative Frequency					

(a) Complete the third row of the table. [2]

(b) Display the results on the cumulative frequency graph below. [2]



(c) Estimate [4]

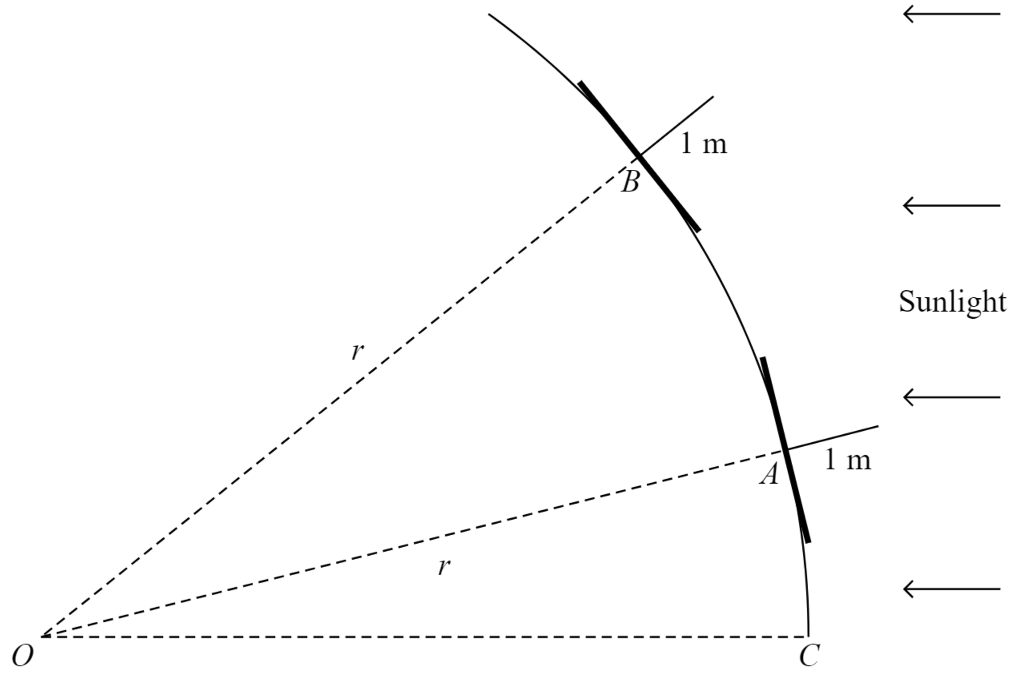
(i) the median number of hours

(ii) the mean number of hours

6. [Maximum points: 8]

It is said that more than 2000 years ago Eratosthenes of Cyrene was able to calculate the size of the earth by measuring the length of the shadow cast by identical vertical poles in two different locations, one directly north of the other, at the same time of day.

The diagram below shows two vertical poles of height 1 m located at points A and B. The ground at each of these points can be considered flat, tangential to the earth. You may assume that the earth is a perfect sphere with radius r and centre O and the rays of light from the sun are parallel.



The pole at point A casts a shadow which is 0.176 metres in length. The pole at point B casts a shadow which is 0.177 metres in length.

(a) Show that $\angle AOC = 9.98183°$. [2]

(b) Determine the value of the following to 5 decimal places. [4]

(i) $\angle BOC$

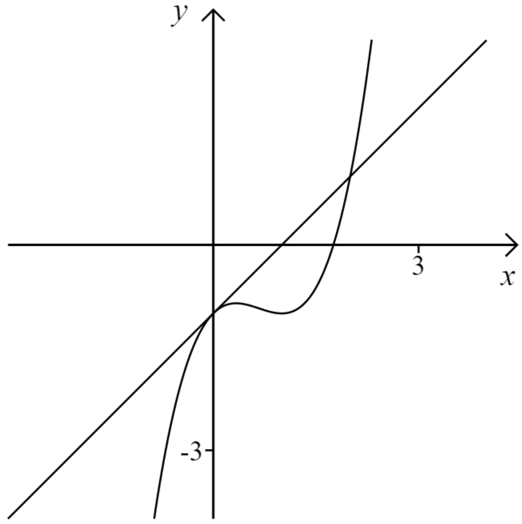
(ii) $\angle BOA$

The distance between points A and B (arc length AB) is equal to 6 km.

(c) Determine the value of r. [2]

7. [Maximum points: 9]

Let $f(x) = x^3 - 2x^2 + x - 1$. The diagram below shows the graph of $y = f(x)$ and the tangent to the graph at $x = 0$.



(a) Find the function $f'(x)$. [2]

(b) Find the equation of the tangent line. Write your answer in the form $y = mx + c$ where $m, c \in \mathbb{Z}$. [3]

(c) Find the coordinates of the point where the tangent line intersects with the curve again. [1]

(d) Find the area of the region completely bound by the curve and the tangent line. [3]

8. [Maximum points: 8]

Consider the series $S_n = 1 + x + x^2 + \cdots + x^{n-2} + x^{n-1}$ where $n \in \mathbb{Z}^+$.

(a) Write down an expression for the value of S_n in terms of x and n. [1]

(b) Hence find $f(x)$ if $x^n - 1 = f(x)S_n$. [1]

(c) Show that [3]

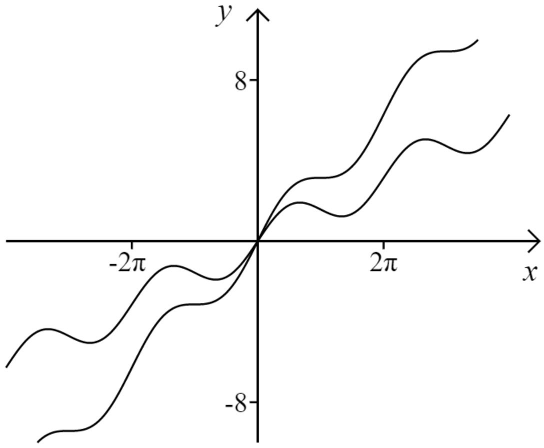
$$2^{ab} - 1 = (2^a - 1)(1 + 2^a + 2^{2a} + \cdots + 2^{a(b-2)} + 2^{a(b-1)})$$

where $a, b \in \mathbb{Z}^+$.

(d) Prove by contradiction that if $2^n - 1$ is prime then n must be prime. [3]

9. [Maximum points: 15]

Let $f(x) = mx + \sin x$ where $m \in \mathbb{R}$. The diagram below shows the graphs of $y = f(x)$ for $m = 0.5$ and 1.



(a) For $-4\pi \le x \le 4\pi$ sketch the graph of $y = f(x)$ when [4]

(i) $m = -0.25$

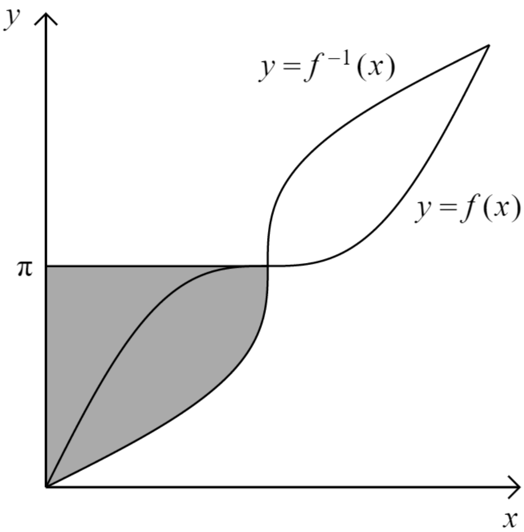
(ii) $m = -1.1$

(b) Find $f'(x)$ writing your answer in terms of m and x. [2]

(c) Find the conditions on m for $f(x)$ to have an inverse function. [5]

(d) When $m = 1$ find $\int_0^{\pi} f(x)\,dx$. [2]

The diagram below shows the graphs of $y = f(x)$ and $y = f^{-1}(x)$ when $m = 1$. The area completely bound by the graph of $y = f^{-1}(x)$, $y = \pi$ and the y-axis is shaded.

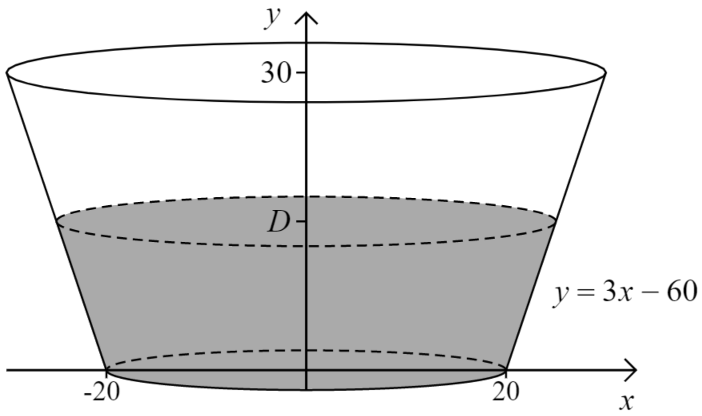


(e) Find the area of the shaded region. [2]

10. [Maximum points: 18]

In this question all units of coordinates are cm.

A bowl is in the shape of the region bound by the graph of $y = 3x - 60$, the y-axis, $y = 0$ and $y = 30$ rotated $360°$ around the y-axis. The bowl is filled with water to a depth of D.



(a) Show that the volume V of water in the bowl is given by [4]

$$V = \frac{\pi(D^3 + 180D^2 + 10800D)}{27}$$

(b) Find the exact volume of water in the bowl when it is completely full. [2]

When bowl is completely full with it begins leaking from the bottom of the bowl at a rate of $50\pi\,\text{cm}^3/\text{sec}$.

(c) Find the length of time it will take the bowl to completely empty. [2]

(d) Hence find the average rate of change of the value of D [2]

(e) One minute after the water starts leaking find [8]

(i) the value of D

(ii) the rate of change of the value of D

11. [Maximum points: 22]

The n^{th} term of a sequence is given by $t_n = 2^{-n}$.

(a) Find the sum of the first three terms. [2]

(b) Find the value of the largest term which is less than 10^{-5}. [4]

Let $f_{n+1}(x) = \sqrt{x f_n(x)}$ where $f_1(x) = \sqrt{x}$ and $x > 1$.

(c) Find the following functions writing your answers in the form x^k where $k \in \mathbb{Q}$. [4]

(i) $f_2(x)$

(ii) $f_3(x)$

(d) If $f_n(x) = x^{g(n)}$ hypothesise an expression for the function $g(n)$. [3]

(e) Prove your answer to (d) using induction. [6]

(f) Evaluate $\log_{\sqrt{x}}\left(f_{10}(x)\right)$ writing your answer as a fraction. [3]

1. (a) $2500\left(1+\dfrac{0.062}{12}\right)^{12} = \$2659.48$ M1A1

(b) We have

$$\frac{2659.48}{2500} = 1.063792$$ M1

So the minimum value of r is 6.38. A1

2. (a) Use normalcdf with lower: 160, upper: 1×10^{99}, μ: 150 and σ: 4.2. M1

The probability is therefore 0.00863. A1

(b)

(i) $0.00863 \times 1000 = 8.63$ A1

(ii) Use binomcdf with trials: 1000, p: 0.00863, x: 10. M1

So $P(X \leq 10) = 0.750$. A1

Therefore $P(X > 10) = 0.250$. A1

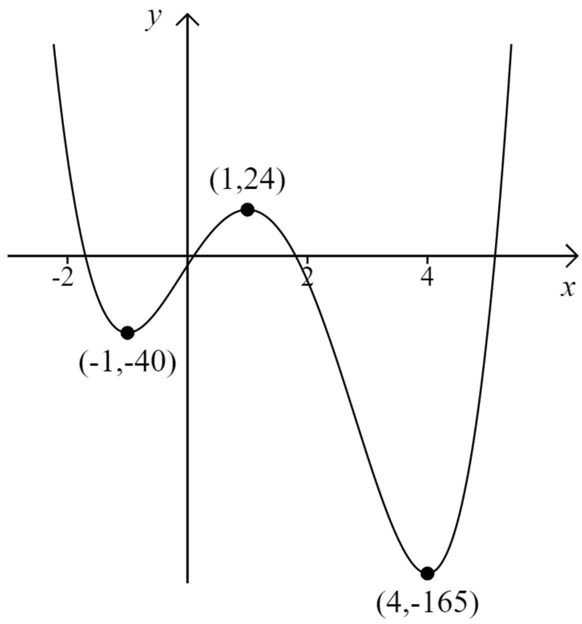
3. The limit is of the form $\frac{0}{0}$ so use l'Hopital's rule. R1M1

$$\lim_{x \to 0} \frac{\cos x - 1}{\sin^2 x} = \lim_{x \to 0} \frac{-\sin x}{2 \sin x \cos x} = \lim_{x \to 0} -\frac{1}{2 \cos x}$$ A1A1

This is equal to

$$-\frac{1}{2 \cos 0} = -\frac{1}{2}$$ M1A1

4. (a) One mark for each correct point A1A1A1



(b)

(i) $x > 24$ or $-165 < k < -40$ A1A1

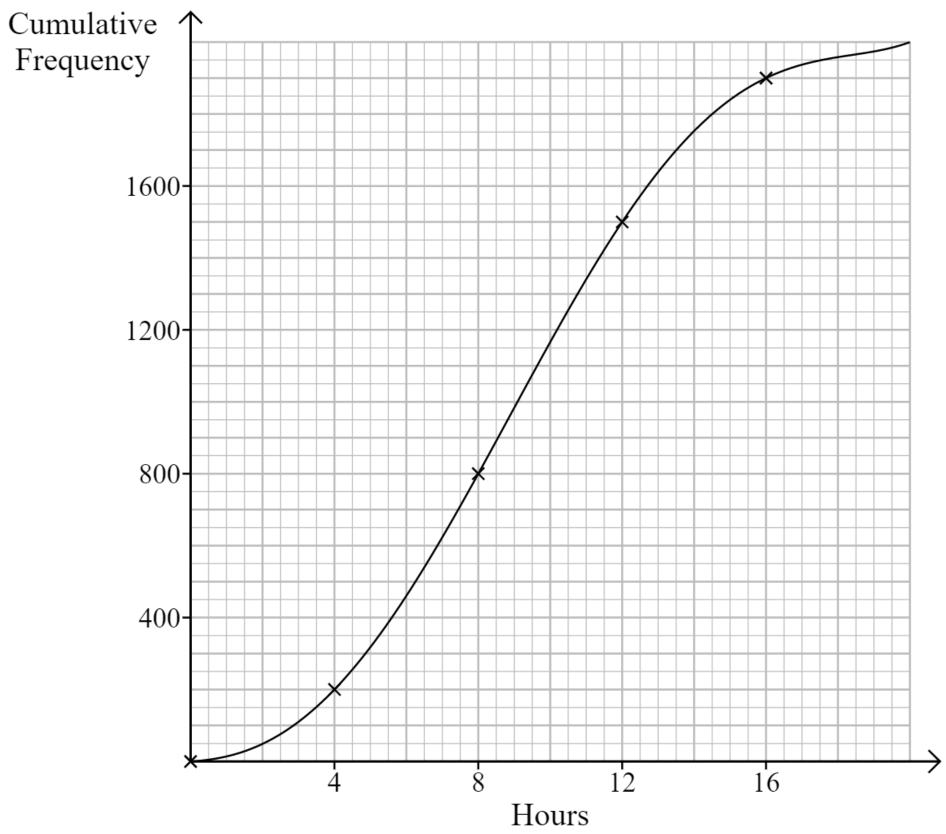
(ii) $-40 < k < 24$ A1

5. (a)

Hours (h)	$0 < h \leq 4$	$4 < h \leq 8$	$8 < h \leq 12$	$12 < h \leq 16$	$16 < h \leq 20$	
Frequency	200	600	700	400	100	
Cumulative Frequency	200	800	1500	1900	2000	A1A1

(b) Use the right-endpoints of the intervals for each x-coordinate and the cumulative frequency for the y-coordinate. A1

Draw a smooth curve through the points. A1



(c)

(i) Consider the value of h when the cumulative frequency is 1000. M1

The median is approximately 9. A1

(ii) $\dfrac{2 \times 200 + 6 \times 600 + 10 \times 700 + 14 \times 400 + 18 \times 100}{1000} = 9.2$ M1A1

6. (a) $\tan^{-1}\left(\dfrac{0.176}{1}\right) = 9.98183°$ M1A1

(b)

(i) $\tan^{-1}\left(\dfrac{0.177}{1}\right) = 10.03739°$ M1A1

(ii) $10.03739 - 9.98183 = 0.05556°$ M1A1

(c) We have

$$\frac{0.05556}{360} \times 2\pi r = 6$$ M1

So $r = 6190$ km. A1

7. (a) $f'(x) = 3x^2 - 4x + 1$ A1A1

(b) We have $f'(0) = 1$. A1

The equation is therefore

$$y - (-1) = 1 \times (x - 0)$$ M1

Giving

$$y = x - 1$$ A1

(c) $(2,1)$ A1

(d) The area is

$$\int_0^2 x - 1 - (x^3 - 2x^2 + x - 1)\, dx = 1.33$$ M1A1A1

8. (a) $\dfrac{x^n - 1}{x - 1}$ A1

(b) $x^n - 1 = (x-1)S_n$ so $f(x) = x - 1$ A1

(c) Replace x with 2^a. We have M1

$$(2^a)^b - 1 = (2^a - 1)(1 + 2^a + (2^a)^2 + \cdots + (2^a)^{b-2} + (2^a)^{b-1})$$ A1

So

$$2^{ab} - 1 = (2^a - 1)(1 + 2^a + 2^{2a} + \cdots + 2^{a(b-2)} + 2^{a(b-1)})$$ A1

(d) If $2^n - 1$ is prime assume that n is composite. This means $n = ab$ where $1 < a,b < n$. M1

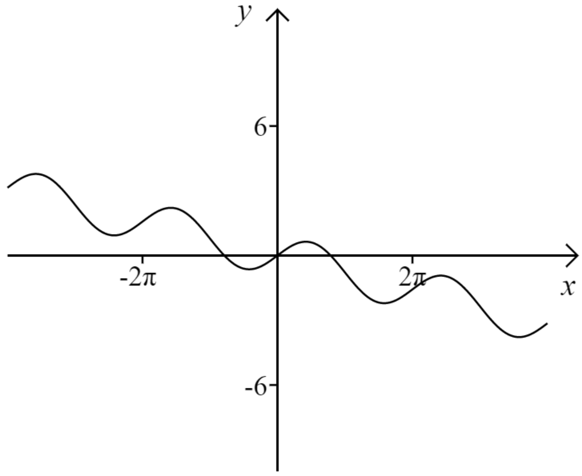
However, by part (c) this means that $2^a - 1$ is a factor of $2^n - 1$ which is a contradiction. R1

So the assumption that n is composite must be false. So n must be prime. A1

9. (a)

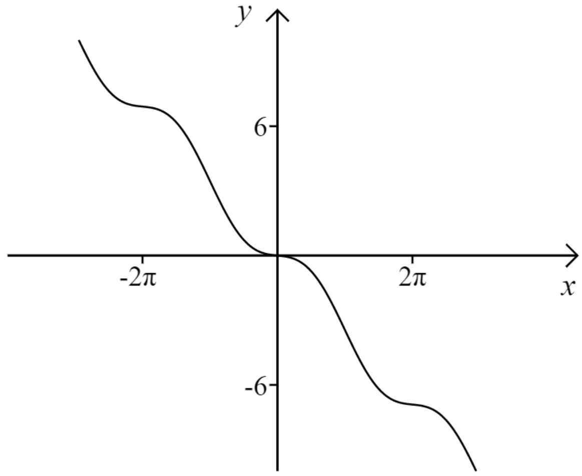
(i) The function has turning points. A1

The shape is approximately correct. A1



(ii) The function is decreasing. A1

The shape is approximately correct. A1



(b) $f'(x) = m + \cos x$ A1A1

(c) The function needs to be an increasing function or a decreasing function. R1

So we need

$$m + \cos x \geq 0$$

or

$$m + \cos x \leq 0 \qquad \text{A1}$$

Since $-1 \leq \cos x \leq 1$ this gives M1

$$m \geq 1 \qquad \text{A1}$$

or

$$m < -1 \qquad \text{A1}$$

(d) Using a GDC $\int_0^{\pi} f(x)\,dx = 6.93$. A1A1

(e) This is equal to $\int_0^{\pi} f(x)\,dx$ so the area is 6.93. R1A1

10. (a) We have

$$V = \pi\int_0^D \left(\frac{y+60}{3}\right)^2 dy = \frac{\pi}{9}\int_0^D y^2 + 120y + 3600\, dy = \frac{\pi}{9}\left[\frac{y^3}{3} + 60y^2 + 3600y\right]_0^D \qquad \text{M1A1A1}$$

This is equal to

$$\frac{\pi}{9} \times \left(\frac{d^3}{3} + 60d^2 + 3600d\right) = \frac{\pi(D^3 + 180D^2 + 10800D)}{27} \qquad \text{A1}$$

(b) $\dfrac{\pi(30^3 + 180 \times 30^2 + 10800 \times 30)}{27} = 19000\pi$ M1A1

(c) It will take $\dfrac{19000\pi}{50\pi} = 380$ seconds for the bowl to empty. M1A1

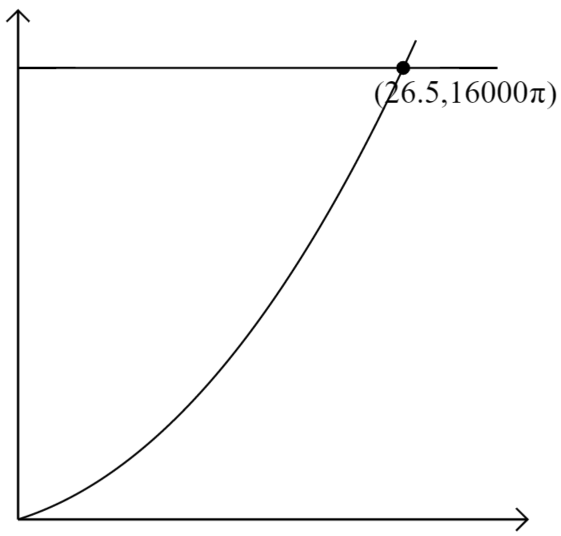
(d) The average rate of change is therefore $-\dfrac{30}{380} = -\dfrac{3}{38}$ cm/s M1A1

(e)

(i) We have

$$19000\pi - 60 \times 50\pi = \frac{(D^3 + 180D^2 + 10800D)}{27} \qquad \text{A1}$$

Solve using any method e.g. using a graph M1



So $D = 26.5$. A1

(ii) We have $\frac{dV}{dt} = -50\pi$. A1

Also

$$\frac{dV}{dD} = \frac{\pi(3D^2 + 360D + 21600)}{27} = \frac{\pi(D^2 + 120D + 7200)}{9}$$ A1

Use $\frac{dV}{dt} = \frac{dV}{dD} \times \frac{dD}{dt}$. M1

So

$$-50\pi = \frac{\pi(26.5^2 + 120 \times 26.5 + 7200)}{9} \times \frac{dD}{dt}$$ A1

Giving

$$\frac{dD}{dt} = -0.0406$$ A1

11. (a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$ M1A1

(b) We have

$$2^{-n} < 10^{-5}$$ A1

So

$$-n\log 2 < \log(10^{-5})$$ M1

Giving

$$n > -\frac{\log(10^{-5})}{\log 2} \approx 16.6$$ A1

So the value is $2^{-17}=\frac{1}{131072}$ A1

(c)

(i) $f_2(x)=\sqrt{x\sqrt{x}}=x^{3/4}$ M1A1

(ii) $f_3(x)=\sqrt{x^{7/4}}=x^{7/8}$ M1A1

(d) Use the geometric series formula to determine the exponent M1

$$g(n)=\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}=1-2^{-n}$$ A1A1

(e) When $n=1$ the exponent is $1-2^{-1}=\frac{1}{2}$. So it is true for $n=1$. A1

Assume it is true for $n=k$. So the exponent is $1-2^{-k}$. A1

For $n=k+1$ the exponent will be $\left(1-2^{-k}+1\right)\times\frac{1}{2}=1-2^{-(k+1)}$. M1A1

So it is true for $n=k+1$. A1

By the principle of mathematical induction it must be true for all $n\in\mathbb{Z}^+$. A1

(f) We have

$$\log_{\sqrt{x}}\left(x^{1-2^{-10}}\right)=\frac{\log_x\left(x^{1-2^{-10}}\right)}{\log_x\sqrt{x}}=2(1-2^{-10})=\frac{2046}{1024}=\frac{1023}{512}$$ M1A1A1